

A NOTE ON Γ_n -ISOMETRIES

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ABSTRACT. In this note we characterize the distinguished boundary of the symmetrized polydisc and thereby develop a model theory for Γ_n -isometries along the lines of [1]. We further prove that for invariant subspaces of Γ_n -isometries, similar to the case $n = 2$ [9], Beurling-Lax-Halmos type representation holds.

1. INTRODUCTION

We denote by \mathbb{D} and $\overline{\mathbb{D}}$ the open and closed unit discs in the complex plane \mathbb{C} . Let $s_i, i \geq 0$, be the *elementary symmetric function* in n variables of degree i , that is, s_i is the sum of all products of i distinct variables z_i so that $s_0 = 1$ and

$$s_i(z_1, \dots, z_n) = \sum_{1 \leq k_1 < k_2 < \dots < k_i \leq n} z_{k_1} \cdots z_{k_i}.$$

For $n \geq 1$, let $\mathbf{s} : \mathbb{C}^n \rightarrow \mathbb{C}^n$ be the function of symmetrization given by the formula

$$\mathbf{s}(z_1, \dots, z_n) = (s_1(z_1, \dots, z_n), \dots, s_n(z_1, \dots, z_n)).$$

The image $\Gamma_n := \mathbf{s}(\overline{\mathbb{D}}^n)$ under the map \mathbf{s} of the unit n -polydisc is known as the *symmetrized n -disc*. The map \mathbf{s} is a proper holomorphic map [8].

Following [1], any commuting n -tuple of operators having Γ_n as a spectral set will be called a Γ_n -contraction. Many of the fundamental results in the theory of contractions have close parallels for Γ_2 -contractions as shown in [1]. In this paper we investigate properties of Γ_n -contractions and we give a model for Γ_n -isometries. As an application, we prove a Beurling-Lax-Halmos type theorem characterizing joint invariant subspaces of a pure Γ_n -isometry. We also indicate how to construct a large class of examples of Γ_n -contractions.

Although a Γ_2 -contraction can be obtained by symmetrizing any pair of commuting contractions [1], it is no longer true that the symmetrization of any n -tuple of commuting contractions will necessarily give rise to a Γ_n -contraction, if $n > 2$ (see Remark 2.12). In fact, the symmetrization of an n -tuple of commuting contractions (T_1, \dots, T_n) is a Γ_n -contraction if and only if (T_1, \dots, T_n) satisfies the analogue of von Neumann's inequality for all symmetric polynomials in n variables (see Proposition 2.13). However, it is shown in [1, Examples 1.7 and 2.3] that not all Γ_2 -contractions are obtained in this way.

We usually denote a typical point of Γ_n by (s_1, \dots, s_n) . We shall also use the notation (S_1, \dots, S_n) for an n -tuple of commuting operators associated in some way with Γ_n . In this paper an *operator* will always be a bounded linear operator on a Hilbert space. The polynomial ring in n variables over the

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field of complex numbers is denoted by $\mathbb{C}[z_1, \dots, z_n]$. Consider a commuting n -tuple (S_1, \dots, S_n) of operators. We say that Γ_n is a *spectral set* for (S_1, \dots, S_n) , or that (S_1, \dots, S_n) is a Γ_n -*contraction*, if, for every polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$,

$$(1.1) \quad \|p(S_1, \dots, S_n)\| \leq \sup_{z \in \Gamma_n} |p(z)| = \|p\|_{\infty, \Gamma_n}.$$

Furthermore, Γ_n is said to be a *complete spectral set* for (S_1, \dots, S_n) , or (S_1, \dots, S_n) to be a *complete Γ_n -contraction*, if, for every matricial polynomial p in n variables,

$$\|p(S_1, \dots, S_n)\| \leq \sup_{z \in \Gamma_n} \|p(z)\|.$$

Here, if S_1, \dots, S_n act on a Hilbert space \mathcal{H} and the matricial polynomial p is given by $p = [p_{ij}]$ of order $m \times \ell$, where each p_{ij} is a scalar polynomial, then $p(S_1, \dots, S_n)$ denotes the operator from \mathcal{H}^ℓ to \mathcal{H}^m with block matrix $[p_{ij}(S_1, \dots, S_n)]$. It is a deep result [1, Theorem 1.5] that a Γ_n -contraction is always a complete Γ_n -contraction and vice versa, for $n = 2$. It is not clear whether a similar result is true for $n > 2$. This will be considered in a future work.

We denote the unit circle by \mathbb{T} . The distinguished boundary of Γ_n , denoted by $b\Gamma_n$, defined to be the Silov boundary of the algebra of functions which are continuous on Γ_n and analytic on the interior of Γ_n , is $\mathbf{s}(\mathbb{T}^n)$ [3, Lemma 8]. We shall use some spaces of vector-valued and operator-valued functions. We recall them following [1]. Let \mathcal{E} be a separable Hilbert space. We denote by $\mathcal{L}(\mathcal{E})$ the space of operators on \mathcal{E} , with the operator norm. Let $H^2(\mathcal{E})$ denote the usual Hardy space of analytic \mathcal{E} -valued functions on \mathbb{D} and $L^2(\mathcal{E})$ the Hilbert space of square integrable \mathcal{E} -valued functions on \mathbb{T} , with their natural inner products. Let $H^\infty \mathcal{L}(\mathcal{E})$ denote the space of bounded analytic $\mathcal{L}(\mathcal{E})$ -valued functions on \mathbb{D} and $L^\infty \mathcal{L}(\mathcal{E})$ the space of bounded measurable $\mathcal{L}(\mathcal{E})$ -valued functions on \mathbb{T} , each with appropriate version of the supremum norm. For $\varphi \in L^\infty \mathcal{L}(\mathcal{E})$ we denote by T_φ the Toeplitz operator with symbol φ , given by

$$T_\varphi f = P_+(\varphi f), \quad f \in H^2(\mathcal{E}),$$

where $P_+ : L^2(\mathcal{E}) \rightarrow H^2(\mathcal{E})$ is the orthogonal projection. In particular T_z is the unilateral shift operator on $H^2(\mathcal{E})$ (the identity function on \mathbb{T} will be denoted by z) and $T_{\bar{z}}$ is the backward shift on $H^2(\mathcal{E})$.

The symmetrization map \mathbf{s} is a proper holomorphic map, $\overline{\mathbb{D}}^n = \mathbf{s}^{-1}(\Gamma_n) = \mathbf{s}^{-1}(\mathbf{s}(\overline{\mathbb{D}}^n))$ and $\overline{\mathbb{D}}^n$ is polynomially convex. Therefore, $\mathbf{s}(\overline{\mathbb{D}}^n) = \Gamma_n$ is polynomially convex by [10, Theorem 1.6.24]. Although we have defined Γ_n -contractions by requiring that the inequality (1.1) holds for all polynomials p in $\mathbb{C}[z_1, \dots, z_n]$, this is equivalent to the definition of Γ_n -contractions by requiring (1.1) to hold for all functions p analytic in a neighbourhood of Γ_n due to polynomial convexity of Γ_n as explained in [1]. As discussed in [1], the subtleties surrounding the various notions of joint spectrum and functional calculus for commuting tuples of operators are not relevant to this paper, simply because of the polynomial convexity of Γ_n .

2. Γ_n AND Γ_n -CONTRACTIONS

Note that $(s_1, \dots, s_n) \in \Gamma_n$ if and only if all the zeros of the polynomial $\sum_{i=0}^n (-1)^{n-i} s_{n-i} z^i$ lie in $\overline{\mathbb{D}}$. This realization of points of Γ_n will be used repeatedly. We state two theorems about location of zeros of polynomials which will be useful in the sequel. For a polynomial $p \in \mathbb{C}[z]$, the derivative of p with respect to z will be denoted by p' .

Theorem 2.1. (*Gauss-Lucas*, [6, page. 22]) *The zeros of the derivative of a polynomial p lie in the convex hull of the zeros of p .*

We recall a definition.

Definition 2.2. A polynomial $p \in \mathbb{C}[z]$ of degree d is called *self-inversive* if $z^d \overline{p(\frac{1}{\bar{z}})} = \omega p(z)$ for some constant $\omega \in \mathbb{C}$ with $|\omega| = 1$.

Theorem 2.3. (*Cohn*, [6, page, 206]) *A necessary and sufficient condition for all the zeros of a polynomial p to lie on the unit circle \mathbb{T} is that p is self-inversive and all the zeros of p' lie in the closed unit disc $\overline{\mathbb{D}}$.*

We shall need characterizations of the distinguished boundary of Γ_n .

Theorem 2.4. *Let $s_i \in \mathbb{C}, i = 1, \dots, n$. The following are equivalent:*

- (i) (s_1, \dots, s_n) is in the distinguished boundary of Γ_n ;
- (ii) $|s_n| = 1, \bar{s}_n s_i = \bar{s}_{n-i}$ and $(\gamma_1 s_1, \dots, \gamma_{n-1} s_{n-1}) \in \Gamma_{n-1}$, where $\gamma_i = \frac{n-i}{n}$ for $i = 1, \dots, n-1$;
- (iii) $(s_1, \dots, s_n) \in \Gamma_n$ and $|s_n| = 1$.

Proof. Throughout this proof, we put $s_0 = 1$. Let (s_1, \dots, s_n) be in the distinguished boundary of Γ_n . By definition, there are $\lambda_i \in \mathbb{T}$ such that

$$(2.1) \quad s_i = \sum_{1 \leq k_1 < \dots < k_i \leq n} \lambda_{k_1} \dots \lambda_{k_i} \text{ for } i = 1, \dots, n.$$

This is equivalent to the fact that the polynomial p , given by

$$(2.2) \quad p(z) = \sum_{i=0}^n (-1)^{n-i} s_{n-i} z^i$$

has all its zeros on \mathbb{T} . Moreover, we clearly have $|s_n| = 1, \bar{s}_n s_i = \bar{s}_{n-i}$ for $i = 1, \dots, n-1$. It follows from Theorem 2.3 that the polynomial

$$(2.3) \quad p'(z) = \sum_{i=1}^n (-1)^{n-i} i s_{n-i} z^{i-1}$$

has all its roots in $\overline{\mathbb{D}}$, which is equivalent to the fact that $(\gamma_1 s_1, \dots, \gamma_{n-1} s_{n-1}) \in \Gamma_{n-1}$, where $\gamma_i = \frac{n-i}{n}$ for $i = 1, \dots, n-1$. Therefore (i) implies (ii).

Conversely, considering the polynomial in Equation (2.2), we observe that

$$z^n \overline{p\left(\frac{1}{\bar{z}}\right)} = \sum_{i=0}^n (-1)^i \bar{s}_i z^i \text{ and } (-1)^n s_n z^n \overline{p\left(\frac{1}{\bar{z}}\right)} = p(z)$$

by the first part of (ii). Therefore p is a self-inversive polynomial. Note that all the roots of p' lies in $\overline{\mathbb{D}}$ as $(\gamma_1 s_1, \dots, \gamma_{n-1} s_{n-1}) \in \Gamma_{n-1}$, where $\gamma_i = \frac{n-i}{n}$ for $i = 1, \dots, n-1$. Thus, it follows from Theorem 2.3 that p has all its roots on \mathbb{T} . This is same as saying that (s_1, \dots, s_n) is in the distinguished boundary of Γ_n .

Clearly, (i) implies (iii). To see the converse, we note that (iii) implies that there exist $\lambda_i \in \mathbb{T}, i = 1, \dots, n$, such that (2.1) holds and $|s_n| = |\lambda_1| \dots |\lambda_n| = 1$. So $\lambda_i \in \mathbb{T}$ for all $i = 1, \dots, n$. It follows from the definition and (2.1) that (s_1, \dots, s_n) is in the distinguished boundary of Γ_n . \square

Remark 2.5. From the Lemma above, similar to the part (4) of [2, Theorem 1.3], one can give expressions for s_j 's, $j = 1, \dots, n$. Clearly, $s_n = \lambda_1 \dots \lambda_n = e^{i\theta}$ for some θ in \mathbb{R} . So $\lambda_n = e^{i\theta} \bar{\lambda}_1 \dots \bar{\lambda}_{n-1}$. Since

$$s_j = \sum_{1 \leq k_1 < k_2 < \dots < k_j \leq n} \lambda_{k_1} \dots \lambda_{k_j} \text{ with } |\lambda_j| = 1$$

for $j = 1, \dots, n$, we have

$$s_j = \sum_{1 \leq k_1 < k_2 < \dots < k_j \leq n} \lambda_{k_1} \dots \lambda_{k_j} = \mu_j + \bar{\mu}_{j-1} \lambda_n$$

where

$$\mu_j = \sum_{1 \leq k_1 < k_2 < \dots < k_j \leq n-1} \lambda_{k_1} \dots \lambda_{k_j}.$$

Since $\lambda_n = e^{i\theta} \bar{\lambda}_1 \dots \bar{\lambda}_{n-1}$ we obtain

$$s_j = \mu_j + \bar{\mu}_{j-1} e^{i\theta} \bar{\lambda}_1 \dots \bar{\lambda}_{n-1} = \mu_j + \mu_{j-1} \bar{\mu}_{n-1} e^{i\theta} = \mu_j + \mu_{n-j} e^{i\theta},$$

since $(\mu_1, \dots, \mu_{n-1}) \in b\Gamma_{n-1}$, $\bar{\mu}_{n-1} \mu_{j-1} = \bar{\mu}_{n-j}$, $j = 1, \dots, n-1$.

Lemma 2.6. *If $(s_1, \dots, s_n) \in \Gamma_n$, then $(\gamma_1 s_1, \dots, \gamma_{n-1} s_{n-1}) \in \Gamma_{n-1}$, where $\gamma_i = \frac{n-i}{n}$ for $i = 1, \dots, n-1$.*

Proof. The hypothesis is equivalent to the fact that the polynomial $p(z) = \sum_{i=0}^n (-1)^{n-i} s_{n-i} z^i$, has all its roots in the closed unit disc $\overline{\mathbb{D}}$. It follows from Theorem 2.1 that the polynomial $p'(z) = \sum_{i=1}^n (-1)^{n-i} i s_{n-i} z^{i-1}$ has all its roots in the closed unit disc as well. Hence we have the desired conclusion. \square

Remark 2.7. Considering the map $\pi : \mathbb{C}^n \longrightarrow \mathbb{C}^{n-1}$ defined by $\pi(z_1, \dots, z_n) = (\gamma_1 z_1, \dots, \gamma_{n-1} z_{n-1})$ the above Lemma can be restated as $\pi(\Gamma_n) \subseteq \Gamma_{n-1}$.

Lemma 2.8. *If (S_1, \dots, S_n) is a Γ_n -contraction, then $(\gamma_1 S_1, \dots, \gamma_{n-1} S_{n-1})$ is a Γ_{n-1} -contraction, where $\gamma_i = \frac{n-i}{n}$ for $i = 1, \dots, n-1$.*

Proof. For $p \in \mathbb{C}[z_1, \dots, z_{n-1}]$, we note that $p \circ \pi \in \mathbb{C}[z_1, \dots, z_n]$ and by hypothesis

$$\begin{aligned} \|p(\gamma_1 S_1, \dots, \gamma_{n-1} S_{n-1})\| &= \|p \circ \pi(S_1, \dots, S_n)\| \\ &\leq \|p \circ \pi\|_{\infty, \Gamma_n} \\ &= \|p\|_{\infty, \pi(\Gamma_n)} \leq \|p\|_{\infty, \Gamma_{n-1}}. \end{aligned}$$

This completes the proof. \square

Lemma 2.9. *If $(s_1, \dots, s_n) \in \Gamma_n$, then $(\alpha + s_1, \alpha s_1 + s_2, \dots, \alpha s_{n-1} + s_n, \alpha s_n) \in \Gamma_{n+1}$ for all α in $\overline{\mathbb{D}}$.*

Proof. If $(s_1, \dots, s_n) \in \Gamma_n$, then it follows from the definition of Γ_n that there are $\lambda_k \in \overline{\mathbb{D}}, k = 1, \dots, n$ such that

$$s_i = \sum_{1 \leq k_1 < k_2 < \dots < k_i \leq n} \lambda_{k_1} \dots \lambda_{k_i}, i = 1, \dots, n.$$

If $\alpha (= \lambda_{n+1}) \in \overline{\mathbb{D}}$, then $(\tilde{s}_1, \dots, \tilde{s}_{n+1}) \in \Gamma_{n+1}$, where

$$\tilde{s}_i = \sum_{1 \leq k_1 < k_2 < \dots < k_i \leq n+1} \lambda_{k_1} \dots \lambda_{k_i}, i = 1, \dots, n+1.$$

Putting $s_0 = 1$ and $s_{n+1} = 0$, we note that $\tilde{s}_i = \alpha s_{i-1} + s_i, i = 1, \dots, n+1$. Therefore we have the desired conclusion. \square

Remark 2.10. For any $\alpha \in \mathbb{C}$, considering the one-to-one map $\pi_\alpha : \mathbb{C}^n \longrightarrow \mathbb{C}^{n+1}$ defined by

$$\pi_\alpha(z_1, \dots, z_n) = (\alpha + z_1, \alpha z_1 + z_2, \dots, \alpha z_{n-1} + z_n, \alpha z_n),$$

the above Lemma can be restated as $\pi_\alpha(\Gamma_n) \subseteq \Gamma_{n+1}$ for all $\alpha \in \overline{\mathbb{D}}$.

Lemma 2.11. *Let (S_1, \dots, S_n) be a Γ_n -contraction, then $(\alpha + S_1, \alpha S_1 + S_2, \dots, \alpha S_{n-1} + S_n, \alpha S_n)$ is a Γ_{n+1} -contraction for all α in $\overline{\mathbb{D}}$.*

Proof. For $p \in \mathbb{C}[z_1, \dots, z_{n+1}]$, we observe that $p \circ \pi_\alpha \in \mathbb{C}[z_1, \dots, z_n]$, so by hypothesis we have

$$\begin{aligned} & \|p(\alpha + S_1, \alpha S_1 + S_2, \dots, \alpha S_{n-1} + S_n, \alpha S_n)\| \\ &= \|p \circ \pi_\alpha(S_1, \dots, S_n)\| \\ &\leq \|p \circ \pi_\alpha\|_{\infty, \Gamma_n} \\ &= \|p\|_{\infty, \pi_\alpha(\Gamma_n)} \\ &\leq \|p\|_{\infty, \Gamma_{n+1}}. \end{aligned}$$

Hence the proof. \square

For an n -tuple $\mathbf{T} = (T_1, \dots, T_n)$ of commuting operators on a Hilbert space \mathcal{H} , let $\mathbf{s}(\mathbf{T}) := (s_1(\mathbf{T}), \dots, s_n(\mathbf{T}))$, we call $\mathbf{s}(\mathbf{T})$, the symmetrization of \mathbf{T} . A polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$ is called *symmetric* if $p(z_\sigma) := p(z_{\sigma(1)}, \dots, z_{\sigma(n)}) = p(\mathbf{z})$ for all $\mathbf{z} \in \mathbb{C}^n$ and $\sigma \in \Sigma_n$, where Σ_n is the symmetric group on n symbols. If $p \in \mathbb{C}[z_1, \dots, z_n]$ is symmetric, then there is a unique $q \in \mathbb{C}[z_1, \dots, z_n]$ such that $p = q \circ \mathbf{s}$, where \mathbf{s} is the symmetrization map [4, Theorem 3.3.1].

Remark 2.12. One may be tempted to conjecture that Lemma above is true when α is replaced by a contraction operator T which commutes with all the S_i 's, $i = 1, \dots, n$. However, this is no longer true. We take $n = 2$ and give an example of a Γ_2 -contraction (S_1, S_2) and a contraction T such that $(T + S_1, TS_1 + S_2, TS_2)$ is not a Γ_3 -contraction. Let (A_1, A_2, A_3) be a tuple of commuting contractions as in Kaijser-Varopoulos [7, Example 5.7]. We take $S_1 = A_1 + A_2, S_2 = A_1 A_2$ and $T = A_3$. Clearly, $(S_1, S_2) = (A_1 + A_2, A_1 A_2)$ is a Γ_2 -contraction due to Ando's inequality. But $(T + S_1, TS_1 + S_2, TS_2) = \mathbf{s}(A_1, A_2, A_3)$ is not a Γ_3 -contraction due to the failure of von Neumann's inequality for more than two commuting contractions. Consider the symmetric polynomial

$$p(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2 - 2z_1 z_2 - 2z_2 z_3 - 2z_3 z_1$$

in [7, Example 5.7]. Taking q to be the polynomial in 3-variables such that $q \circ \mathbf{s} = p$, where \mathbf{s} is the symmetrization map, one observes that Γ_3 cannot be a spectral set for $\mathbf{s}(A_1, A_2, A_3)$.

Proposition 2.13. *The symmetrization of an n -tuple of commuting contractions (T_1, \dots, T_n) is a Γ_n -contraction if and only if (T_1, \dots, T_n) satisfies the analogue of von Neumann's inequality for all symmetric polynomials in $\mathbb{C}[z_1, \dots, z_n]$.*

Proof. Let $\mathbf{T} = (T_1, \dots, T_n)$ be a commuting n -tuple of contractions satisfying the analogue of von Neumann's inequality for all symmetric polynomials $\mathbb{C}[z_1, \dots, z_n]$. We show that $\mathbf{s}(\mathbf{T}) = (s_1(\mathbf{T}), \dots, s_n(\mathbf{T}))$ is a Γ_n -contraction. For a polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$, we note that

$$\|p(s_1(\mathbf{T}), \dots, s_n(\mathbf{T}))\| = \|p \circ \mathbf{s}(\mathbf{T})\| \leq \|p \circ \mathbf{s}\|_{\infty, \overline{\mathbb{D}}^n} = \|p\|_{\infty, \mathbf{s}(\overline{\mathbb{D}}^n)},$$

since $p \circ \mathbf{s}$ is a symmetric polynomial, the above inequality holds by hypothesis, and it shows that $\Gamma_n = \mathbf{s}(\overline{\mathbb{D}}^n)$ is a spectral set for $(s_1(\mathbf{T}), \dots, s_n(\mathbf{T}))$.

Conversely, let $\mathbf{T} = (T_1, \dots, T_n)$ be a tuple of commuting contractions such that $\mathbf{s}(\mathbf{T})$ is a Γ_n -contraction and $q \in \mathbb{C}[z_1, \dots, z_n]$ be symmetric. So there is a $p \in \mathbb{C}[z_1, \dots, z_n]$ such that $p \circ \mathbf{s} = q$. By hypothesis, we have

$$\|q(\mathbf{T})\| = \|p(\mathbf{s}(\mathbf{T}))\| \leq \|p\|_{\infty, \Gamma_n} = \|p \circ \mathbf{s}\|_{\infty, \overline{\mathbb{D}}^n} = \|q\|_{\infty, \overline{\mathbb{D}}^n}.$$

□

Next, we give a straightforward generalization of the Lemma 3.2 in [2].

Proposition 2.14. *Let $\mathbf{T} = (T_1, \dots, T_n)$ be a commuting n -tuple of contractions on a Hilbert space \mathcal{H} satisfying the analogue of von Neumann's inequality for all symmetric polynomials in $\mathbb{C}[z_1, \dots, z_n]$. Let \mathcal{M} be an invariant subspace for $s_i(\mathbf{T}), i = 1, \dots, n$. Then $(s_1(\mathbf{T})|_{\mathcal{M}}, \dots, s_n(\mathbf{T})|_{\mathcal{M}})$ is a Γ_n -contraction on the Hilbert space \mathcal{M} .*

Proof. For a polynomial $p \in \mathbb{C}[z_1, \dots, z_n]$, we note that

$$\begin{aligned} \|p(s_1(\mathbf{T})|_{\mathcal{M}}, \dots, s_n(\mathbf{T})|_{\mathcal{M}})\| &= \|p(s_1(\mathbf{T}), \dots, s_n(\mathbf{T}))|_{\mathcal{M}}\| \\ &\leq \|p(s_1(\mathbf{T}), \dots, s_n(\mathbf{T}))\| \\ &\leq \|p\|_{\infty, \Gamma_n}, \end{aligned}$$

since $\mathbf{s}(\mathbf{T}) = (s_1(\mathbf{T}), \dots, s_n(\mathbf{T}))$ is a Γ_n -contraction on \mathcal{H} , the last inequality holds by the previous Proposition. Hence we have the desired conclusion. □

Remark 2.15.

1. It is clear from the proof of the Proposition above that if (S_1, \dots, S_n) is a Γ_n -contraction on a Hilbert space \mathcal{H} and \mathcal{M} is a common invariant subspace for $S_i, i = 1, \dots, n$, then $(S_1|_{\mathcal{M}}, \dots, S_n|_{\mathcal{M}})$ is a Γ_n -contraction on \mathcal{M} .
2. As an immediate consequence of Proposition 2.13, we observe that the symmetrizations of the classes of n -tuples of commuting contractions discussed in [5] give rise to a large class of Γ_n -contractions.
3. It is shown in [2] that applying the Lemma 3.2, a large class of Γ_2 -contractions can be constructed. In an analogous way, examples of Γ_n -contractions can be constructed for any integer $n > 2$ applying Proposition 2.14.
4. From Theorem 3.2 in [1], it is clear that all Γ_2 -contractions are obtained by applying the Lemma 3.2 in [2] as described in the same paper. However, for $n > 3$, it is not clear whether all Γ_n -contractions have similar realizations.

3. Γ_n -UNITARY AND Γ_n -ISOMETRIES

We start with the following obvious generalizations of definitions in [1].

Definition 3.1. Let S_1, \dots, S_n be commuting operators on a Hilbert space \mathcal{H} . We say that (S_1, \dots, S_n) is

- (i) a Γ_n -unitary if $S_i, i = 1, \dots, n$ are normal operators and the joint spectrum $\sigma(S_1, \dots, S_n)$ of (S_1, \dots, S_n) is contained in the distinguished boundary of Γ_n .
- (ii) a Γ_n -isometry if there exists a Hilbert space \mathcal{K} containing \mathcal{H} and a Γ_n -unitary $(\tilde{S}_1, \dots, \tilde{S}_n)$ on \mathcal{K} such that \mathcal{H} is invariant for $\tilde{S}_i, i = 1, \dots, n$ and $S_i = \tilde{S}_i|_{\mathcal{H}}, i = 1, \dots, n$.

- (iii) a Γ_n -co-isometry if (S_1^*, \dots, S_n^*) is a Γ -isometry.
- (iv) a pure Γ_n -isometry if (S_1, \dots, S_n) is a Γ_n -isometry and S_n is a pure isometry.

The proof of the following theorem works along the lines of Agler and Young[1].

Theorem 3.2. *Let S_i , $i = 1, \dots, n$, be commuting operators on a Hilbert space \mathcal{H} . The following are equivalent:*

- (i) (S_1, \dots, S_n) is a Γ_n -unitary;
- (ii) $S_n^* S_n = I = S_n S_n^*$, $S_n^* S_i = S_{n-i}^*$ and $(\gamma_1 S_1, \dots, \gamma_{n-1} S_{n-1})$ is a Γ_{n-1} -contraction, where $\gamma_i = \frac{n-i}{n}$ for $i = 1, \dots, n-1$;
- (iii) there exist commuting unitary operators U_i for $i = 1, \dots, n$ on \mathcal{H} such that

$$S_i = \sum_{1 \leq k_1 < \dots < k_i \leq n} U_{k_1} \dots U_{k_i} \text{ for } i = 1, \dots, n.$$

Proof. Suppose (i) holds. Let (S_1, \dots, S_n) be a Γ_n -unitary. By the spectral theorem for commuting normal operators, there exists a spectral measure $M(\cdot)$ on $\sigma(S_1, \dots, S_n)$ such that

$$S_i = \int_{\sigma(S_1, \dots, S_n)} s_i(z) M(dz), \quad i = 1, \dots, n,$$

where s_1, \dots, s_n are the co-ordinate functions on \mathbb{C}^n . Let τ be a measurable right inverse of the restriction of \mathbf{s} to \mathbb{T}^n , so that τ maps distinguished boundary of Γ_n to \mathbb{T}^n . Let $\tau = (\tau_1, \dots, \tau_n)$ and

$$U_i = \int_{\sigma(S_1, \dots, S_n)} \tau_i(z) M(dz), \quad i = 1, \dots, n.$$

Clearly U_1, \dots, U_n are commuting unitaries on \mathcal{H} and

$$\begin{aligned} \sum_{1 \leq k_1 < \dots < k_i \leq n} U_{k_1} \dots U_{k_i} &= \sum_{1 \leq k_1 < \dots < k_i \leq n} \int_{\sigma(S_1, \dots, S_n)} \tau_{k_1}(z) \dots \tau_{k_i}(z) M(dz) \\ &= \int_{\sigma(S_1, \dots, S_n)} s_i(z) M(dz) \\ &= S_i, \end{aligned}$$

for $i = 1, \dots, n$. Hence (i) implies (iii).

Suppose (iii) holds. Then $S_n^* S_n = I = S_n S_n^*$ and $S_n^* S_i = S_{n-i}^*$, $i = 1, \dots, n$ follow immediately. Moreover, since (S_1, \dots, S_n) is a Γ_n -contraction, we have from Lemma 2.8 that $(\gamma_1 S_1, \dots, \gamma_{n-1} S_{n-1})$ is a Γ_{n-1} -contraction, where $\gamma_i = \frac{n-i}{n}$ for $i = 1, \dots, n-1$. Hence (iii) implies (ii).

Suppose (ii) holds. Since S_n is normal, by Fuglede's theorem $S_{n-i}^* S_{n-i} = S_n^* S_i S_i^* S_n = S_i S_i^*$. Now as S_i 's commute, $S_{n-i}^* S_{n-i} = S_n^* S_i S_{n-i} = S_n^* S_{n-i} S_i = S_i^* S_i$ and we have each of S_i , $i = 1, \dots, n$ is normal. So the unital C^* -algebra $C^*(S_1, \dots, S_n)$ generated by S_1, \dots, S_n is commutative and by Gelfand-Naimark's theorem is $*$ -isometrically isomorphic to $C(\sigma(S_1, \dots, S_n))$. Let $\hat{S}_1, \dots, \hat{S}_n$ be the images of S_1, \dots, S_n under the Gelfand map. By definition, for an arbitrary point $\mathbf{z} = (s_1, \dots, s_n)$ in $\sigma(S_1, \dots, S_n)$, $\hat{S}_i(\mathbf{z}) = s_i$ for $i = 1, \dots, n$. By properties of the Gelfand map and hypothesis we have,

$$\overline{\hat{S}_n(\mathbf{z})} \hat{S}_n(\mathbf{z}) = 1 = \hat{S}_n(\mathbf{z}) \overline{\hat{S}_n(\mathbf{z})} \text{ and } \overline{\hat{S}_n(\mathbf{z})} \hat{S}_i(\mathbf{z}) = \overline{\hat{S}_{n-i}(\mathbf{z})} \text{ for } \mathbf{z} \text{ in } \sigma(S_1, \dots, S_n).$$

Thus we obtain $|s_n| = 1$, $\bar{s}_n s_i = \bar{s}_{n-i}$. Now $(\gamma_1 S_1, \dots, \gamma_{n-1} S_{n-1})$ is a Γ_{n-1} -contraction implies

$$\|p(\gamma_1 S_1, \dots, \gamma_{n-1} S_{n-1})\| \leq \|p\|_{\infty, \Gamma_{n-1}}$$

which is equivalent to $\|p\|_{\infty, \Gamma_{n-1}}^2 - p(\gamma_1 S_1, \dots, \gamma_{n-1} S_{n-1})^* p(\gamma_1 S_1, \dots, \gamma_{n-1} S_{n-1})$ is positive. Applying Gelfand transform we have

$$\|p\|_{\infty, \Gamma_{n-1}}^2 - p(\gamma_1 \hat{S}_1(z), \dots, \gamma_{n-1} \hat{S}_{n-1}(z))^* p(\gamma_1 \hat{S}_1(z), \dots, \gamma_{n-1} \hat{S}_{n-1}(z)) \geq 0$$

for z in $\sigma(S_1, \dots, S_n)$. This shows $(\gamma_1 s_1, \dots, \gamma_{n-1} s_{n-1})$ is in the polynomially convex hull of Γ_{n-1} . Since Γ_{n-1} is polynomially convex, $(\gamma_1 s_1, \dots, \gamma_{n-1} s_{n-1})$ is in Γ_{n-1} . Therefore by Theorem 2.4 (s_1, \dots, s_n) is in the distinguished boundary of Γ_n and hence $\sigma(S_1, \dots, S_n) \subset b\Gamma_n$. This proves (ii) implies (i). \square

It is not a priori clear whether unitarity of S_n would imply that (S_1, \dots, S_n) to be Γ_n -unitary. However, the following is true.

Lemma 3.3. *Let $\mathbf{T} = (T_1, \dots, T_n)$ be a tuple of commuting contractions on a Hilbert space \mathcal{H} . If $s_n(\mathbf{T}) = (\prod_{i=1}^n T_i)$ is a unitary, then $\mathbf{s}(\mathbf{T})$ is a Γ_n -unitary.*

Proof. Note that each of the T_i , $i = 1, \dots, n$ is invertible as they are commuting with each other and their product is unitary. Since each T_i is a contraction, we have $\|T_i^{-1}\| \geq 1$. First we show that $\|T_i^{-1}\| = 1$ for all $i = 1, \dots, n$. If not, then for some k in $\{1, \dots, n\}$ we have $\|T_k^{-1}\| > 1$. So, there exist a y in \mathcal{H} such that $\|T_k^{-1}y\| > \|y\|$. Let $(\prod_{i \neq k} T_i)y = x$. Thus

$$\|s_n(\mathbf{T})^{-1}\| \geq \frac{\|s_n(\mathbf{T})^{-1}x\|}{\|x\|} = \frac{\|T_k^{-1}y\|}{\|y\|} \cdot \frac{\|y\|}{\|(\prod_{i \neq k} T_i)y\|} > 1$$

is a contradiction as by hypothesis $s_n(\mathbf{T})^{-1}$ is also a unitary. Next we show that each T_i is isometry, that is, $T_i^* T_i = I$, $i = 1, \dots, n$. Suppose for some ℓ in $\{1, \dots, n\}$, $T_\ell^* T_\ell \neq I$. There exists nonzero y in \mathcal{H} such that $(I - T_\ell^* T_\ell)y \neq 0$. Since T_ℓ is a contraction, $\|y\|^2 - \|T_\ell y\|^2 = \langle (I - T_\ell^* T_\ell)y, y \rangle = \|(I - T_\ell^* T_\ell)^{\frac{1}{2}} y\|^2 > 0$. But this shows, $\|T_\ell^{-1}\| > 1$ which is a contradiction. Thus $T_i^* T_i = I$, $i = 1, \dots, n$. Applying the same trick to $\mathbf{T}^* = (T_1^*, \dots, T_n^*)$, we see that each T_i is a unitary. Hence by part (iii) of the Theorem above, $\mathbf{s}(\mathbf{T})$ is a Γ_n -unitary. \square

The following Lemma will be useful in characterizing pure Γ_n -isometry.

Lemma 3.4. *Let Φ_1, \dots, Φ_n be functions in $L^\infty \mathcal{L}(\mathcal{E})$ and M_{Φ_i} , $i = 1, \dots, n$, denotes the corresponding multiplication operator on $L^2(\mathcal{E})$. Then $(M_{\Phi_1}, \dots, M_{\Phi_n})$ is a Γ_n contraction if and only if $(\Phi_1(z), \dots, \Phi_n(z))$ is a Γ_n contraction for all z in \mathbb{T} .*

Proof. Note that for $\|M_\Psi\| = \|\Psi\|_\infty$ for $\Psi \in L^\infty \mathcal{L}(\mathcal{E})$. By definition, $(M_{\Phi_1}, \dots, M_{\Phi_n})$ is a Γ_n contraction if and only if

$$(3.1) \quad \|p(M_{\Phi_1}, \dots, M_{\Phi_n})\| \leq \|p\|_{\infty, \Gamma_n},$$

for all polynomials $p \in \mathbb{C}[z_1, \dots, z_n]$. Since $p(M_{\Phi_1}, \dots, M_{\Phi_n}) = M_{p(\Phi_1, \dots, \Phi_n)}$ and $\|M_\Psi\| = \|\Psi\|_\infty := \sup_{z \in \mathbb{T}} \|\Psi(z)\|$ for $\Psi \in L^\infty \mathcal{L}(\mathcal{E})$, we have (3.1) is true if and only if

$$\|p(\Phi_1(z), \dots, \Phi_n(z))\| \leq \|p\|_{\infty, \Gamma_n},$$

which completes the proof. \square

Remark 3.5. An interesting case in the Lemma above, is when $\dim \mathcal{E} = 1$. In this case, the n -tuple $(\Phi_1(z), \dots, \Phi_n(z))$ is a Γ_n -contraction means that $(\Phi_1(z), \dots, \Phi_n(z))$ is in Γ_n which is true as Γ_n is polynomially convex.

Theorem 3.6. *Let S_i , $i = 1, \dots, n$, be commuting operators on a Hilbert space \mathcal{H} . Then (S_1, \dots, S_n) is a pure Γ_n -isometry if and only if there exist a separable Hilbert space \mathcal{E} and a unitary operator $U : \mathcal{H} \rightarrow H^2(\mathcal{E})$ and functions $\Phi_1, \dots, \Phi_{n-1}$ in $H^\infty \mathcal{L}(\mathcal{E})$ and operators $A_i \in \mathcal{L}(\mathcal{E})$ $i = 1, \dots, n-1$ such that*

- (i) $S_i = U^* M_{\Phi_i} U$, $i = 1, \dots, n-1$, $S_n = U^* M_z U$;
- (ii) $(\gamma_1 \Phi_1(z), \dots, \gamma_{n-1} \Phi_{n-1}(z))$ is a Γ_{n-1} -contraction for all z in \mathbb{T} , where $\gamma_i = \frac{n-i}{n}$ for $i = 1, \dots, n-1$;
- (iii) $\Phi_i(z) = A_i + A_{n-i}^* z$ for $1 \leq i \leq n-1$;
- (iv) $[A_i, A_j] = 0$ and $[A_i, A_{n-j}^*] = [A_j, A_{n-i}^*]$ for $1 \leq i, j \leq n-1$, where $[P, Q] = PQ - QP$ for two operators P, Q .

Proof. Suppose (S_1, \dots, S_n) is a pure Γ_n -isometry. By definition, there exist a Hilbert space \mathcal{K} and a Γ_n -unitary $(\tilde{S}_1, \dots, \tilde{S}_n)$ such that $\mathcal{H} \subset \mathcal{K}$ is a common invariant subspace of \tilde{S}_i 's and $S_i = \tilde{S}_i|_{\mathcal{H}}$, $i = 1, \dots, n$. From Theorem 3.2, it follows that $\tilde{S}_n^* \tilde{S}_n = I$ and $\tilde{S}_n^* \tilde{S}_i = \tilde{S}_{n-i}^*$, $1 \leq i, j \leq n-1$. By compression of \tilde{S}_i to \mathcal{H} , we have

$$S_n^* S_n = I \text{ and } S_n^* S_i = S_{n-i}^*, 1 \leq i \leq n-1.$$

Since S_n is a pure isometry and \mathcal{H} is separable, there exist a unitary operator $U : \mathcal{H} \rightarrow H^2(\mathcal{E})$, for some separable Hilbert space \mathcal{E} , such that $S_n = U^* M_z U$, where M_z is the shift operator on $H^2(\mathcal{E})$. Since S_i 's commute with S_n , there exist $\Phi_i \in H^\infty \mathcal{L}(\mathcal{E})$ such that $S_i = U^* M_{\Phi_i} U$ for $1 \leq i \leq n-1$. As (S_1, \dots, S_n) is a Γ_n -contraction, from Lemma 2.8 and Lemma 3.4 part (ii) follows. The relations $S_n^* S_i = S_{n-i}^*$ yield

$$M_{\bar{z}} M_{\Phi_i} = M_{\Phi_{n-i}}^*, 1 \leq i \leq n-1.$$

Let $\Phi(z) = \sum_{n \geq 0} C_n z^n$, $\Psi(z) = \sum_{n \geq 0} D_n z^n$ be in $H^\infty \mathcal{L}(\mathcal{E})$. Then $M_{\bar{z}} M_\Phi = M_\Psi^*$ implies

$$C_0 \bar{z} + C_1 + \sum_{n \geq 2} C_n z^{n-1} = D_0^* + D_1^* \bar{z} + \sum_{n \geq 2} D_n^* \bar{z}^n$$

for all $z \in \mathbb{T}$ and by comparing coefficients, we get

$$C_0 = D_1^* \text{ and } C_1 = D_0^*.$$

This gives that each $\Phi_i(z)$ is of the form $A_i + B_i z$ for some $A_i, B_i \in \mathcal{L}(\mathcal{E})$, where $A_i = B_{n-i}^*$ for $1 \leq i \leq n-1$, which is part (iii). Now since S_i 's commutes, we have $M_{\Phi_i} M_{\Phi_j} = M_{\Phi_j} M_{\Phi_i}$ and consequently

$$(A_i + A_{n-i}^* z)(A_j + A_{n-j}^* z) = (A_j + A_{n-j}^* z)(A_i + A_{n-i}^* z)$$

for $1 \leq i, j \leq n-1$ and for all $z \in \mathbb{T}$. Comparing the constant term and the coefficients of z , we get part (iv).

Conversely, suppose (S_1, \dots, S_n) be the n -tuple satisfying conditions (i) to (iv). Consider the n -tuple $(M_{\Phi_1}, \dots, M_{\Phi_{n-1}}, M_z)$ of multiplication operators on $L^2(\mathcal{E})$ with symbols $\Phi_1, \dots, \Phi_{n-1}, z$ respectively. From condition (iv), it follows that M_{Φ_i} 's commutes with each other. Part of condition (iii) shows that $M_{\bar{z}} M_{\Phi_i} = M_{\Phi_{n-i}}^*$ by repeating calculations similar to above. Thus, it is easy to see from part (ii) of Theorem 3.2, that $(M_{\Phi_1}, \dots, M_{\Phi_{n-1}}, M_z)$ is a Γ_n -unitary and so is $(U^* M_{\Phi_1} U, \dots, U^* M_{\Phi_{n-1}} U, U^* M_z U)$. Now, S_i 's, $1 \leq i \leq n$, are the restrictions to the common invariant subspace $H^2(\mathcal{E})$ of $(M_{\Phi_1}, \dots, M_{\Phi_{n-1}}, M_z)$ and hence (S_1, \dots, S_n) is a Γ_n -isometry. Since S_n is a shift, (S_1, \dots, S_n) is a pure Γ_n -isometry. \square

Next, we obtain characterization for Γ_n -isometry analogous to the Wold decomposition in terms of Γ_n -unitaries and pure Γ_n -isometries using Theorem 3.2 and Theorem 3.6. We prove this along the way of Agler and Young [1] and will use the following Lemma [1, Lemma 2.5].

Lemma 3.7. *Let U and V be a unitary and a pure isometry on Hilbert space $\mathcal{H}_1, \mathcal{H}_2$ respectively, and let $T : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ be an operator such that $TU = VT$. Then $T = 0$.*

Theorem 3.8. *Let $S_i, i = 1, \dots, n$, be commuting operators on a Hilbert space \mathcal{H} . The following are equivalent:*

- (i) (S_1, \dots, S_n) is a Γ_n -isometry;
- (ii) *There exists a orthogonal decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ into common reducing subspaces of S_i 's, $i = 1, \dots, n$ such that $(S_1|_{\mathcal{H}_1}, \dots, S_n|_{\mathcal{H}_1})$ is a Γ_n -unitary and $(S_1|_{\mathcal{H}_2}, \dots, S_n|_{\mathcal{H}_2})$ is a pure Γ_n -isometry;*
- (iii) $S_n^* S_n = I$, $S_n^* S_i = S_{n-i}^*$ and $(\gamma_1 S_1, \dots, \gamma_{n-1} S_{n-1})$ is a Γ_{n-1} -contraction, where $\gamma_i = \frac{n-i}{n}$ for $i = 1, \dots, n-1$.

Proof. Suppose (i) holds. By definition, there exists $(\tilde{S}_1, \dots, \tilde{S}_n)$ a Γ_n -unitary on \mathcal{K} containing \mathcal{H} such that \mathcal{H} is a invariant subspace of \tilde{S}_i 's and S_i 's are restrictions of \tilde{S}_i 's to \mathcal{H} . From Theorem 3.2, it follows that \tilde{S}_i 's are satisfying the relations: $\tilde{S}_n^* \tilde{S}_n = I$, $\tilde{S}_n^* \tilde{S}_i = \tilde{S}_{n-i}^*$ and $(\gamma_1 \tilde{S}_1, \dots, \gamma_{n-1} \tilde{S}_{n-1})$ is a Γ_{n-1} -contraction, where $\gamma_i = \frac{n-i}{n}$ for $i = 1, \dots, n-1$. Compressing to the common invariant subspace \mathcal{H} and by part (1) of remark 2.15, we obtain

$$S_n^* S_n = I, S_n^* S_i = S_{n-i}^*$$

and $(\gamma_1 S_1, \dots, \gamma_{n-1} S_{n-1})$ is a Γ_{n-1} -contraction, where $\gamma_i = \frac{n-i}{n}$ for $i = 1, \dots, n-1$. Thus (i) implies (iii).

Suppose (iii) holds. By Wold decomposition, we may write $S_n = U \oplus V$ on $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ where $\mathcal{H}_1, \mathcal{H}_2$ are reducing subspaces for S_n , U is unitary and V is pure isometry. Let us write

$$S_i = \begin{bmatrix} S_{11}^{(i)} & S_{12}^{(i)} \\ S_{21}^{(i)} & S_{22}^{(i)} \end{bmatrix}$$

with respect to this decomposition, where $S_{jk}^{(i)}$ is a bounded operator from \mathcal{H}_k to \mathcal{H}_j . Since $S_n S_i = S_i S_n$, we have

$$\begin{bmatrix} U S_{11}^{(i)} & U S_{12}^{(i)} \\ V S_{21}^{(i)} & V S_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} S_{11}^{(i)} U & S_{12}^{(i)} V \\ S_{21}^{(i)} U & S_{22}^{(i)} V \end{bmatrix}, i = 1, \dots, n-1.$$

Thus, $S_{21}^{(i)} U = V S_{21}^{(i)}$ and hence by Lemma 3.7, $S_{21}^{(i)} = 0$, $i = 1, \dots, n-1$. Now $S_n^* S_i = S_{n-i}^*$ gives

$$\begin{bmatrix} U^* S_{11}^{(i)} & U^* S_{12}^{(i)} \\ 0 & V^* S_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} S_{11}^{(n-i)*} & 0 \\ S_{12}^{(n-i)*} & S_{22}^{(n-i)*} \end{bmatrix}, i = 1, \dots, n-1.$$

It follows that $S_{12}^{(i)} = 0$, $i = 1, \dots, n-1$. So $\mathcal{H}_1, \mathcal{H}_2$ are common reducing subspace for S_1, \dots, S_n . From the matrix equation above, we have $U^* S_{11}^{(i)} = S_{11}^{(n-i)*}$, $i = 1, \dots, n-1$. Thus by part (i) of the remark 2.15 and part (ii) of Theorem 3.2, it follows that $(S_{11}^{(1)}, \dots, S_{11}^{(n-1)}, U)$ is a Γ_n -unitary.

We now require to show that $(S_{22}^{(1)}, \dots, S_{22}^{(n-1)}, V)$ is a pure Γ_n -isometry. Since V is a pure isometry \mathcal{H} is separable, we can identify it with the shift operator M_z on the space of vector valued functions $H^2(\mathcal{E})$, for some separable Hilbert space \mathcal{E} . Since $S_{22}^{(i)}$'s commute with V , there exists

$\Phi_i \in H^\infty \mathcal{L}(\mathcal{E})$ such that $S_{22}^{(i)}$ can be identified with M_{Φ_i} for $1 \leq i \leq n-1$. As (S_1, \dots, S_n) is a Γ_n -contraction, from Lemma 2.8 and Lemma 3.4, it follows that $(\gamma_1 S_{22}^{(1)}, \dots, \gamma_{n-1} S_{22}^{(n-1)})$ is a Γ_{n-1} -contraction, where $\gamma_i = \frac{n-i}{n}$ for $i = 1, \dots, n-1$. The relations $V^* S_{22}^{(i)} = S_{22}^{(n-i)*}$ yield

$$M_{\bar{z}} M_{\Phi_i} = M_{\Phi_{n-i}}^*, \quad 1 \leq i \leq n-1.$$

Calculations similar to the first part of the proof of Theorem 3.6, we get each $\Phi_i(z)$ is of the form $A_i + A_{n-i}^* z$ for some A_i 's in $\mathcal{L}(\mathcal{E})$, $i = 1, \dots, n-1$. Now since $S_{22}^{(i)}$'s commutes, we have

$$[A_i, A_j] = 0 \text{ and } [A_i, A_{n-j}^*] = [A_j, A_{n-i}^*] \text{ for } 1 \leq i, j \leq n-1.$$

Hence, by Theorem 3.6, $(S_1|_{\mathcal{H}_2}, \dots, S_n|_{\mathcal{H}_2})$ is a pure Γ_n -isometry. Thus (iii) implies (ii).

It is easy to see that (ii) implies (i). \square

Corollary 3.9. *Let S_i , $i = 1, \dots, n$, be commuting operators on a Hilbert space \mathcal{H} . (S_1, \dots, S_n) is a Γ_n -co-isometry if and only if*

$$S_n S_n^* = I, \quad S_n S_i^* = S_{n-i}$$

and $(\gamma_1 S_1^, \dots, \gamma_{n-1} S_{n-1}^*)$ is a Γ_{n-1} -contraction, where $\gamma_i = \frac{n-i}{n}$ for $i = 1, \dots, n-1$.*

4. CHARACTERIZATION OF INVARIANT SUBSPACES FOR Γ_n -ISOMETRIES

This section is devoted to characterize the joint invariant subspaces of pure Γ_n -isometries. Similar characterizations for pure Γ_2 -isometries appears in [9].

In the light of Theorem 3.6, we start with a characterization of pure Γ_n -isometries in terms of the parameters associated with them. For simplicity of notation, let (M_{Φ}, M_z) denote the n -tuple of multiplication operators $(M_{\Phi_1}, \dots, M_{\Phi_{n-1}}, M_z)$ on $H^2(\mathcal{E})$, where $\Phi_i \in H^\infty \mathcal{L}(\mathcal{E})$, $i = 1, \dots, n-1$. Throughout this section, we will be using the canonical identification of $H^2(\mathcal{E})$ with $H^2 \otimes \mathcal{E}$ by the map $z^n \xi \mapsto z^n \otimes \xi$, where $n \in \mathbb{N} \cup \{0\}$ and $\xi \in \mathcal{E}$, whenever necessary.

Theorem 4.1. *Let $\Phi_i(z) = A_i + A_{n-i}^* z$ and $\tilde{\Phi}_i(z) = \tilde{A}_i + \tilde{A}_{n-i}^* z$ be in $H^\infty \mathcal{L}(\mathcal{E})$ for some $A_i \in \mathcal{L}(\mathcal{E})$ and $\tilde{A}_i \in \mathcal{L}(\mathcal{F})$ respectively, $i = 1, \dots, n-1$. Then the n -tuple (M_{Φ}, M_z) on $H^2(\mathcal{E})$ is unitarily equivalent to the n -tuple $(M_{\tilde{\Phi}}, M_z)$ on $H^2(\mathcal{F})$ if and only if the $(n-1)$ -tuples (A_1, \dots, A_{n-1}) and $(\tilde{A}_1, \dots, \tilde{A}_{n-1})$ are unitarily equivalent.*

Proof. Suppose the n -tuple (M_{Φ}, M_z) on $H^2(\mathcal{E})$ is unitarily equivalent to the n -tuple $(M_{\tilde{\Phi}}, M_z)$ on $H^2(\mathcal{F})$. We can identify the map Φ_i (similarly $\tilde{\Phi}_i$) by $I_{H^2} \otimes A_i + M_z \otimes A_{n-i}^*$. So, there exist a unitary $U : H^2 \otimes \mathcal{E} \rightarrow H^2 \otimes \mathcal{F}$ such that

$$(4.1) \quad U(I_{H^2} \otimes A_i + M_z \otimes A_{n-i}^*)U^* = I_{H^2} \otimes \tilde{A}_i + M_z \otimes \tilde{A}_{n-i}^*, \quad i = 1, \dots, n-1,$$

and

$$(4.2) \quad U(M_z \otimes I_{\mathcal{E}})U^* = M_z \otimes I_{\mathcal{F}}.$$

From equation (4.2), it follows that there exists a unitary $\tilde{U} : \mathcal{E} \rightarrow \mathcal{F}$ such that $U = I_{H^2} \otimes \tilde{U}$. Consequently, the equation (4.1) can be written as

$$\tilde{U} A_i \tilde{U}^* + \tilde{U} A_{n-i}^* z \tilde{U}^* z = \tilde{A}_i + \tilde{A}_{n-i}^* z, \quad i = 1, \dots, n-1,$$

for all $z \in \mathbb{T}$. Hence comparing the coefficients, we obtain $\tilde{U} A_i \tilde{U}^* = \tilde{A}_i$, $i = 1, \dots, n-1$ which completes the proof in forward direction.

Conversely, suppose there exist a unitary $\tilde{U} : \mathcal{E} \rightarrow \mathcal{F}$ that intertwines A_i and \tilde{A}_i , that is, $\tilde{U} A_i \tilde{U}^* = \tilde{A}_i$ for each $i = 1, \dots, n-1$. Let $U : H^2 \otimes \mathcal{E} \rightarrow H^2 \otimes \mathcal{F}$ be the map defined by

$U = I_{H^2} \otimes \tilde{U}$. Clearly, U is a unitary and from the computations similar to above, it is easy to see that U intertwines M_{Φ_i} with $M_{\tilde{\Phi}_i}$ for each $i = 1, \dots, n-1$, and M_z . This completes the proof. \square

In the following corollary we express the above theorem in terms of pure Γ_n -isometries.

Corollary 4.2. *Let (S_1, \dots, S_n) and $(\tilde{S}_1, \dots, \tilde{S}_n)$ be a pair of pure Γ_n -isometries. Then (S_1, \dots, S_n) and $(\tilde{S}_1, \dots, \tilde{S}_n)$ are unitarily equivalent if and only if the $(n-1)$ -tuples $(S_{n-i}^* - S_i S_n^*)_{i=1}^{n-1}$ and $(\tilde{S}_{n-i}^* - \tilde{S}_i \tilde{S}_n^*)_{i=1}^{n-1}$ are unitary equivalent.*

Proof. Let the pure Γ_n -isometry (S_1, \dots, S_n) is equivalent to (M_{Φ}, M_z) where $\Phi_i(z) = A_i + A_{n-i}^* z$ be in $H^\infty \mathcal{L}(\mathcal{E})$ for some $A_i \in \mathcal{L}(\mathcal{E})$, $i = 1, \dots, n-1$ satisfying conditions (ii) to (iv) in Theorem 3.2. Clearly the $(n-1)$ -tuple $(S_{n-i}^* - S_i S_n^*)_{i=1}^{n-1}$ is equivalent to $(M_{\Phi_{n-i}}^* - M_{\Phi_i} M_z^*)_{i=1}^{n-1}$. Note that with the canonical identification we have

$$\begin{aligned} M_{\Phi_{n-i}}^* - M_{\Phi_i} M_z^* &= (I_{H^2} \otimes A_{n-i} + M_z \otimes A_i^*)^* - (I_{H^2} \otimes A_i + M_z \otimes A_{n-i}^*)(M_z^* \otimes I_{\mathcal{E}}) \\ &= (I_{H^2} - M_z M_z^*) \otimes A_{n-i}^* \\ &= P_{\mathbb{C}} \otimes A_{n-i}^* \end{aligned}$$

where $P_{\mathbb{C}}$ is the orthogonal projection from H^2 to the scalars in H^2 . Thus it follows for Theorem 4.1 that unitary equivalence of the n -tuple (M_{Φ}, M_z) is determined by unitary equivalence of the $(n-1)$ -tuple $(M_{\Phi_{n-i}}^* - M_{\Phi_i} M_z^*)_{i=1}^{n-1}$, the unitary equivalence of (S_1, \dots, S_n) is determined by $(S_{n-i}^* - S_i S_n^*)_{i=1}^{n-1}$. This completes the proof. \square

Characterization of invariant subspaces of Γ_n -isometries essentially boils down to that of pure Γ_n -isometries due to Wold type decomposition in Theorem 3.8 which is again same as characterizing invariant subspace for the associated model space obtained in Theorem 3.6. The following theorem discusses this issue.

A closed subspace $\mathcal{M} \neq \{0\}$ of $H^2(\mathcal{E})$ is said to be (M_{Φ}, M_z) -invariant if \mathcal{M} is invariant under M_{Φ_i} and M_z for all $i = 1, \dots, n-1$.

Let $\mathcal{M} \neq \{0\}$ be a closed subspace of $H^2(\mathcal{E}_*)$. It follows from Beurling-Lax-Halmos theorem that \mathcal{M} is invariant under M_z if and only if there exists a Hilbert space \mathcal{E} and an inner function $\Theta \in H^\infty \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ (Θ is an isometry almost everywhere on \mathbb{T}) such that

$$\mathcal{M} = M_{\Theta} H^2(\mathcal{E}).$$

Theorem 4.3. *Let $\mathcal{M} \neq 0$ be a closed subspace of $H^2(\mathcal{E}_*)$ and $\Phi_i, i = 1, \dots, n-1$ be in $H^\infty \mathcal{L}(\mathcal{E}_*)$ such that (M_{Φ}, M_z) is a pure Γ_n -isometry on $H^2(\mathcal{E}_*)$. Then \mathcal{M} is a (M_{Φ}, M_z) -invariant subspace if and only if there exist $\Psi_i, i = 1, \dots, n-1$ in $H^\infty \mathcal{L}(\mathcal{E})$ such that (M_{Ψ}, M_z) is a pure Γ_n -isometry on $H^2(\mathcal{E})$ and*

$$\Phi_i \Theta = \Theta \Psi_i, \quad i = 1, \dots, n-1,$$

where $\Theta \in H^\infty \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ is the Beurling-Lax-Halmos representation of \mathcal{M} .

Proof. We will prove only the forward direction as the other part is easy to see. Let \mathcal{M} is invariant under (M_{Φ}, M_z) . Thus, in particular, \mathcal{M} is invariant under M_z and hence the Beurling-Lax-Halmos representation of \mathcal{M} is $\Theta H^2(\mathcal{E})$ where $\Theta \in H^\infty \mathcal{L}(\mathcal{E}, \mathcal{E}_*)$ is an inner multiplier. We also have $M_{\Phi_i} \Theta H^2(\mathcal{E}) \subseteq \Theta H^2(\mathcal{E})$ for each $i = 1, \dots, n-1$. Thus, there exist Ψ_i 's, $i = 1, \dots, n-1$ in $H^\infty \mathcal{L}(\mathcal{E})$ such that $M_{\Phi_i} M_{\Theta} = M_{\Theta} M_{\Psi_i}$, that is, $\Phi_i \Theta = \Theta \Psi_i, i = 1, \dots, n-1$. Since Φ_i 's commute, we

have $M_\Theta M_{\Psi_i} M_{\Psi_j} = M_{\Phi_i} M_{\Phi_j} M_\Theta = M_{\Phi_j} M_{\Phi_i} M_\Theta = M_\Theta M_{\Psi_j} M_{\Psi_i}$ and hence $\Psi_i \Psi_j = \Psi_j \Psi_i$, $i, j = 1, \dots, n-1$. Furthermore, we have

$$p(M_{\Phi_1}, \dots, M_{\Phi_{n-1}}) M_\Theta = M_\Theta p(M_{\Psi_1}, \dots, M_{\Psi_{n-1}})$$

for all polynomials $p \in \mathbb{C}[z_1, \dots, z_{n-1}]$. Therefore,

$$\|p(\gamma_1 M_{\Psi_1}, \dots, \gamma_{n-1} M_{\Psi_{n-1}})\| \leq \|M_\Theta^* p(\gamma_1 M_{\Phi_1}, \dots, \gamma_{n-1} M_{\Phi_{n-1}}) M_\Theta\| \leq \|p\|_{\infty, \Gamma_{n-1}},$$

for all polynomials $p \in \mathbb{C}[z_1, \dots, z_{n-1}]$ and $\gamma_i = \frac{n-i}{n}$ for $i = 1, \dots, n-1$. Using Theorem 3.6, we also note that

$$M_{\bar{z}} M_{\Psi_i} = M_{\bar{z}} M_\Theta^* M_{\Phi_i} M_\Theta = M_\Theta^* M_{\bar{z}} M_{\Phi_i} M_\Theta = M_\Theta^* M_{\Phi_{n-i}}^* M_\Theta = M_{\Psi_{n-i}}^*, \quad i = 1, \dots, n-1.$$

From the observations made above and by Theorem 3.6, it follows that (M_Ψ, M_z) is a pure Γ_n -isometry on $H^2(\mathcal{E})$ and this completes the proof. \square

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